

Harmonic nets in metric spaces

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1 Introduction

In this paper, we consider harmonic maps from weighted graphs into metric spaces. This can be considered as a generalization of geodesic lines in Riemannian manifolds. Our considerations are based on the simple observation that a geodesic considered as a map $\gamma : [0, 1] \rightarrow N$ from the unit interval to the Riemannian manifold N and parametrized proportionally to arclength is characterized by the property that for all sufficiently close $0 \leq a < b \leq 1$, $\gamma(\frac{a+b}{2})$ is the unique midpoint of $\gamma(a)$ and $\gamma(b)$, that is,

$$\gamma\left(\frac{a+b}{2}\right) = \operatorname{argmin}_{q \in N} (d^2(\gamma(a), q) + d^2(\gamma(b), q)). \quad (1)$$

This leads us to represent a geodesic as a string of points in N each of which is the midpoint of its two neighbors. At the same time, this allows us flexible refinements, that is, we can insert additional points in the string as midpoints of consecutive ones already present. For that, it is useful to also consider the following slight generalization of (1)

$$\gamma(ta + (1-t)b) = \operatorname{argmin}_{q \in N} (td^2(\gamma(a), q) + (1-t)d^2(\gamma(b), q)) \quad (2)$$

($0 < t < 1$).

A midpoint is a center of gravity of two points. In a Riemannian manifold, such centers of gravity exist locally uniquely, that is, when the points whose center is to be constructed are sufficiently close. Globally, uniqueness need not be true. Therefore, we may need to localize in the image.

It is then clear how to conceptualize a harmonic map from a weighted graph into N . We simply require that the images of the nodes of the graph are appropriately weighted centers of gravity of their neighbors. Here, in order to localize in the image, we might need to refine the graph by subdividing edges.

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Harmonic maps from graphs into compact Riemannian manifolds were studied in [8]. Our approach, however, naturally leads to a generalization to metric space targets that locally admit such unique centers of gravity. This class of metric spaces includes the important class of Alexandrov spaces with upper curvature bounds, see [1] as a systematic reference.

Thus, in this paper, we show the existence of harmonic maps from weighted graphs into such metric spaces, shortly called harmonic nets. (When we have a space with non-positive curvature in the sense of Alexandrov or Busemann, this result is contained in a general existence result for harmonic maps of the first author, see [4, 6].) The proof is not difficult. It is based on the iterative replacement of image points by the centers of gravity of the images of their neighbors, following the strategy described in [7], together with suitable adaptive refinements to keep the constructions local. Here it is important that the domain, that is, our graph, can be treated as a one-dimensional object. While two dimensions represent a border line case, in higher dimensions, general constructions of harmonic maps are only possible when the target space possesses non-positive curvature. The reason for that is that the energy functional we are employing is quadratic, and therefore the scaling behavior is different in dimensions 1, 2, and greater than 2. Therefore, the essential features of our scheme are local uniqueness and the scaling property of our functionals.

Our constructions possess certain similarities with some schemes employed in numerical analysis, like the standard difference scheme for the numerical solution of the Laplace equation or adaptive refinements in multigrid methods. A key conceptual feature of our approach is that we systematically exploit local uniqueness of solutions and that we need to make explicit only the images of a discrete set of points because then all other images are implicitly determined by that local uniqueness. Therefore, as in good numerical schemes, we never have to work out or store more information than we actually need.

2 Geometric concepts

We let (N, d) be a complete metric space. For abbreviation, we usually simply write N in place of (N, d) , the metric $d(\cdot, \cdot)$ being implicitly understood. We say that N admits refinements if for any $p, q \in N$, there exists some $m \in N$ with

$$d(m, p) = d(m, q) = \frac{1}{2}d(p, q). \quad (3)$$

We also call such an m a midpoint of p and q .

Definition 2.1. Suppose that N admits refinements. We define the radius $r(N)$ of unique refinement as the largest $r \in [0, \infty]$ with the property that for any two $p, q \in N$ with $d(p, q) \leq 2r$, their refinement (midpoint) $m = m(p, q)$ is unique.

More generally, we say that $q \in N$ is a center of gravity of the finitely many points $p_1, \dots, p_n \in N$ with weights w_1, \dots, w_n (> 0) if

$$q = \operatorname{argmin}_p \sum_{j=1}^n w_j^2 d^2(p, p_j). \quad (4)$$

We define the convexity radius $c(N)$ as the largest $c \in [0, \infty]$ with the property that whenever p, p_1, \dots, p_n are points in N with $d(p, p_i) \leq c$ for $i = 1, \dots, n$, then the center of gravity of p_1, \dots, p_n (with any positive weights) is unique. Since a midpoint of two points is their center of gravity when they are given equal weights, we have obviously $0 \leq c(N) \leq r(N)$.

We let Γ be a finite weighted graph with vertex set I and edge set E where each $e \in E$ has a weight $w(e) > 0$. We say that two vertices i, j are neighbors if they are connected by an edge. Thus, for our purposes, a graph is a discrete set I together with a symmetric neighborhood relation \sim and (symmetric) weights $w(i, j) = w(e)$ for any two neighboring vertices i, j (i.e. $i \sim j$) connected by the edge e .

We define the refinement Γ_r of Γ as the graph with vertex set $I \cup E$ with $i \in I$ and $e \in E$ being neighbors when $i \in e$ in Γ ; there are no other pairs of neighbors in Γ_r . The weight of such a pair of neighbors is $w(i, e) = \sqrt{2}w(e)$ where the latter is the weight of the edge e in Γ . The edge set of Γ_r then is obvious. We can then also define successive refinements of Γ_r .

A map f from Γ to N assigns to every $i \in I$ some point $p = f(i)$ in N .

We define the **energy** of such an $f : \Gamma \rightarrow N$ as

$$E(f) = \sum_{i \in I} E_i(f), \quad (5)$$

$$\text{with } E_i(f) = \sum_{j \in I; i \sim j} w^2(i, j) d^2(f(i), f(j)). \quad (6)$$

In particular, for $i \sim j$,

$$d^2(f(i), f(j)) \leq \frac{1}{w^2(i, j)} E_i(f). \quad (7)$$

We say that the map f is **harmonic** if for all $i \in I$, $f(i)$ is a center of gravity of the points $f(j)$, $j \sim i$, with weights $w_j = w^2(i, j)$.

3 Characterization by angles in tangent cones

The above concepts of refinement and center of gravity find their natural place in the context of Alexandrov's metric spaces. For a systematic development of this theory that we shall use in this section, see [1].

These spaces enjoy particular properties when their (Alexandrov) curvature is bounded from above. It is part of the definition of such a space of curvature bounded from above that any two sufficiently close points can be joined by a shortest geodesic which then is in fact unique and depends continuously on these endpoints. (We may also parametrize it by arclength – and call it an arclength geodesic – if convenient.) Some of the general notions in the theory, however, do not need the assumption of an upper curvature bound. That assumption then is rather employed to derive geometric properties of the objects defined in the theory.

An important concept here is the tangent cone of a metric space at a point. Let (N, d) be a metric space, $\gamma_1, \gamma_2 : [0, \varepsilon] \rightarrow (N, d)$ be arclength geodesics emanating from a point $P \in N$.

Consider points $Q \in \gamma_1, R \in \gamma_2$ different from P . An (upper) angle $\theta(\gamma_1, \gamma_2)$ between γ_1 and γ_2 is defined as

$$\cos \theta(\gamma_1, \gamma_2) := \overline{\lim}_{Q, R \rightarrow P} \frac{d^2(P, Q) + d^2(P, R) - d^2(Q, R)}{2d(P, R)d(P, Q)}.$$

We have the following characterization of the angle between γ_1 and γ_2 (see [2], [II.1-II.3]): let (N, d) be a metric space whose curvature is bounded from above by a constant $K \geq 0$. Then

$$\cos \theta(\gamma_1, \gamma_2) = \lim_{s \rightarrow 0} \frac{d(P, \gamma_2(\varepsilon)) - d(\gamma_1(s), \gamma_2(\varepsilon))}{s},$$

provided in case $K > 0$ that ε is less than the diameter of the comparison model space of constant curvature K .

A geodesic curve γ starting at a point $P \in N$ has a direction if $\theta(\gamma, \gamma) = 0$ and two curves have the same direction if the angle between them is equal to zero. This is an equivalence relation on the space of curves starting from the same point $P \in N$ and the completion of the set of equivalence classes (endowed with the distance induced by the angle) is called the space of directions $\Omega_P(N)$ of N at the point P .

The tangent cone $T_P N$ of (N, d) at a point $P \in N$ is the cone over the space of directions, namely $T_P N = (\Omega_P(N) \times \mathbb{R}_+) / (\Omega_P(N) \times \{0\})$.

We will denote a tangent element by $[\gamma, x]$, where $\gamma \in \Omega_P(N)$, $x \geq 0$ and elements $[\gamma, 0]$ are identified with the origin O_P of $T_P N$.

The distance d in N induces on $T_P N$ a distance function \tilde{d}_P defined by

$$\tilde{d}_P^2([\gamma_1, x_1], [\gamma_2, x_2]) = \begin{cases} x_1^2 + x_2^2 - 2x_1x_2 \cos \theta(\gamma_1, \gamma_2) & \text{if } \theta(\gamma_1, \gamma_2) < \pi \\ x_1 + x_2 & \text{if } \theta(\gamma_1, \gamma_2) \geq \pi \end{cases}$$

For those $[\gamma, x]$ for which we can find a unique geodesic from P with direction γ that can be extended up to distance x , we define that point as the exponential image of $[\gamma, x]$. The inverse of this exponential map, the projection map from the subset of N where it is defined to the tangent cone $T_P N$ is denoted by π_P . In the case of (simply connected, complete) nonpositively curved metric spaces, it is well known that the map π_P is defined everywhere, distance non-increasing and distance preserving in the radial direction (see [10]).

The following important result has been proved by Nikolaev [9]:

Lemma 3.1. *Let (N, d) a metric space of curvature $\leq K$, with $K \geq 0$. Then the tangent cone at a point of N is a space of non-positive curvature in the sense of Alexandrov.*

Let P, Q, R be points in N and $Q_s \equiv (1-s)P + sQ$ the point on a distance realizing geodesic joining P and Q with distance $s \cdot d(P, Q)$ from P .

We have the following Taylor expansions:

$$d^2(Q_s, R) = d^2(P, R) - 2sd(P, R) \cos \theta_P(Q, R) + a(s) \quad , \quad \text{with } \lim_{s \rightarrow 0} \frac{a(s)}{s} = 0$$

$$\tilde{d}_P^2(\pi_P(Q_s), \pi_P(R)) = d^2(P, R) - 2sd(P, R) \cos \theta_P(Q, R) + b(s) \quad , \quad \text{with } \lim_{s \rightarrow 0} \frac{b(s)}{s} = 0 \quad ,$$

where $\theta_P(Q, R)$ denotes the angle subtended by Q and R at P .

We recall the following result concerning harmonic maps (see [3, 10]):

Proposition 3.1. *Let $f : \Gamma \longrightarrow (N, d)$ be an harmonic map, then:*

(i) : $\forall i \in I$, $\pi_{f(i)}(f(i))$ minimizes the function $\sum_{j \sim i} w^2(i, j) \tilde{d}_{f(i)}^2(\cdot, \pi_{f(i)} f(j))$

in $T_{f(i)} N$.

(ii) : $\forall i \in I$, $\sum_{j \sim i} w^2(i, j) \langle V, \pi_{f(i)} f(j) \rangle \leq 0$, $\forall V \in T_{f(i)} N$,

where $\langle \cdot, \cdot \rangle$ denotes the inner product defined on $T_{f(i)} N$ by:

$$\langle [\gamma_1, x_1], [\gamma_2, x_2] \rangle = x_1 x_2 \cos \theta(\gamma_1, \gamma_2)$$

(iii) : $\forall i \in I$, the barycenter in $T_{f(i)}$ of the points $(\pi_{f(i)} f(j))_{j \sim i}$ with weights $(\frac{w^2(i, j)}{w(i)})_{j \sim i}$ coincides with the origin $O_i := \pi_{f(i)} f(i)$ of $T_{f(i)} N$, where $w(i) = \sum_{j \sim i} w^2(i, j)$.

The inequality (ii) in the above proposition will be interpreted as the critical condition for harmonic nets.

4 Refining maps

If N admits refinements, we can construct a refinement $f_r : \Gamma_r \rightarrow N$ of a map $f : \Gamma \rightarrow N$ by assigning to every edge e connecting i and j in Γ a midpoint of $f(i)$ and $f(j)$. We observe that for each $i \in \Gamma$, we have

$$E_i(f_r) = \frac{1}{2}E_i(f) \quad (8)$$

where on the left hand side, i is considered as an element of Γ_r . Also,

$$\sum_{i \in I} E_i(f_r) = \sum_{e \in E} E_e(f_r) = \frac{1}{2}E(f_r) \quad (9)$$

by symmetry, where we consider the is and es as vertices of Γ_r . In particular, we have from (8), (9)

Lemma 4.1.

$$E(f_r) = E(f). \quad (10)$$

If $f : \Gamma \rightarrow N$ is harmonic then so is its refinement f_r .

The converse holds when distances between images are sufficiently small, that is, when midpoints between the images of neighbors are unique.

From (7) and (10), we conclude that by performing sufficiently many successive refinements, we may assume that all distances between the images of any two neighboring vertices are smaller than some prescribed $\epsilon > 0$, for example smaller than $r(N)$ or $c(N)$ when that quantity is positive.

5 Homotopy classes

For the present purposes, we write $r(f)$ and $r(\Gamma)$ instead of f_r and Γ_r because we wish to consider the refinement as an operation that can be iterated. For example, $r^2(\Gamma) = (\Gamma_r)_r$ is obtained as the refinement of Γ_r . A refinable map $f : \Gamma \rightarrow N$ then is considered as a collection of iteratively refined maps $r^n(f) : r^n(\Gamma) \rightarrow N$ for $n \in \mathbb{N}$.

Assume now that the refinement radius $r(N) > 0$. We say that two maps $f_1, f_2 : \Gamma \rightarrow N$ are geodesically close if for every $i \in \Gamma$, $d(f_1(i), f_2(i)) \leq 2r(N)$, that is, the images of i under f_1 and f_2 have a unique midpoint. A refinement of the pair f_1, f_2 then is defined to be the triple $f_1, f_{1,2}, f_2$ where $f_{1,2}$ is the midpoint map of f_1 and f_2 , that is, for every $i \in \Gamma$, $f_{1,2}(i)$ is the midpoint of $f_1(i)$ and $f_2(i)$.

We say that two refinable maps $f, g : \Gamma \rightarrow N$ are geometrically homotopic if there exists a sequence $f_0 = f, f_1, f_2, \dots, f_A = g$ for some $A \in \mathbb{N}$ of refinable maps such that for any $n \in \mathbb{N}$ and any $1 \leq j \leq A$, the maps $r^n(f_{j-1})$ and $r^n(f_j)$ are geodesically close. This sequence can again be refined by putting in midpoint maps between consecutive sequence elements.

Geometric homotopy is an equivalence relation, and the equivalence classes are called geometric homotopy classes of refinable maps from Γ to N .

6 Construction of harmonic nets

We assume that N admits centers of gravity. By subdividing suitable edges of Γ as above, we may assume that Γ is bipartite, that is, its vertex set is a disjoint union $I = I_1 \cup I_2$ such that all the neighbors of any point in one of those subsets are contained in the other one. On the space $C = C(\Gamma, N)$ of maps $f : \Gamma \rightarrow N$, we define maps $\rho_\alpha : C \rightarrow C$, $\alpha = 1, 2$ with $\rho_\alpha(f)$ being the map obtained from $f : \Gamma \rightarrow N$ by replacing the image of every $f(i)$ for $i \in I_\alpha$ by a center of gravity of the $f(j)$ for $j \sim i$. As long as the centers of gravity are not unique, we need to make choices here, but in the situation where $c(N) > 0$, we can assume that Γ has been sufficiently refined (depending on an upper bound E for the energy of f) so that the images $f(j)$ of the neighbors of any $i \in \Gamma$ possess a unique center of gravity. (This follows from (7) and the fact that the edge weights get multiplied by a factor of $\sqrt{2}$, that is, become larger, under each refinement.) In that case, the maps $\rho_\alpha(f)$ are unambiguously defined for all f with $E(f) \leq E$. ρ_α decreases (or, more precisely, does not increase) the energy density $E_i(f)$ for all points in I_α , but not necessarily for those in the complement of I_α . Nevertheless, since by symmetry $\sum_{i \in I_1} E_i(f) = \sum_{i \in I_2} E_i(f) = \frac{1}{2}E(f)$ (see (9)), we have

Lemma 6.1.

$$E(\rho_\alpha(f)) \leq E(f) \quad (11)$$

for all f .

Moreover

Lemma 6.2.

$$E(\rho_2(\rho_1(f))) = E(f) \quad (12)$$

if and only if f is harmonic.

Theorem 6.1. *Let (N, d) be a compact metric space that admits centers of gravity. Let Γ be a finite weighted graph. Then, for any map $f : \Gamma \rightarrow N$, the iterations $f_n := (\rho_2 \rho_1)^n f$ contain a subsequence converging to a harmonic map.*

Proof. Since N is compact, we can find some sequence $\nu(n)$ of positive integers going to infinity for which $f_{\nu(n)}(i)$ converges to some point $f_0(i) \in N$ for every vertex i of the finite graph Γ . We have

$$f_{\nu(n+1)} = (\rho_2 \rho_1)^{\mu(n)} f_{\nu(n)} \text{ for some } \mu(n) \in \mathbb{N}. \quad (13)$$

Since the metric d behaves continuously under convergence (since it defines the topology of N), we have

$$E(f_0) = \lim_{n \rightarrow \infty} E(f_{\nu(n)}). \quad (14)$$

But then also

$$\begin{aligned} \lim E(f_{\nu(n+1)}) &= \lim E((\rho_2 \rho_1)^{\mu(n)} f_{\nu(n)}) \\ &\leq \lim E(f_{\nu(n)}), \text{ by Lemma 6.1 because of } \mu(n) \geq 1 \\ &= E(f_0). \end{aligned}$$

Thus, equality has to hold throughout. Moreover, $\rho_2\rho_1 f_{\nu(n)}$ converges to $\rho_2\rho_1 f_0$, and so

$$\begin{aligned} E(\rho_2\rho_1 f_0) &= \lim E((\rho_2\rho_1)^{\mu(n)+1} f_{\nu(n)}) \text{ as before} \\ &= E(f_0) \text{ from the preceding observation.} \end{aligned}$$

Lemma 6.2 then implies that f_0 is harmonic. \square

The assumption of the theorem that the space N admits centers of gravity is satisfied when N has an upper curvature bound. For $k \in \mathbb{R}$, we denote by D_k the diameter of the n -dimensional, complete, simply connected model space with constant sectional curvature k . We then have the following result from Alexandrov's theory (see [1]):

Lemma 6.3. *Let X be an Alexandrov space with curvature bounded above by k . Then, for every $x \in X$, there exists a positive number $R_x \in (0, \frac{D_k}{2}]$ such that the closed metric ball centered at x and with radius R_x , $\overline{B}(x, R_x)$, is a convex subset in X .*

Remark: When N has non-positive curvature in the sense of Alexandrov or Busemann, our theorem is contained in a general theorem of the first author (see [4]) and when N is a compact Riemannian manifold, it follows from the fact that any homotopy class contains at least one harmonic map (see [8]). Since distances of neighboring image points are controlled by the energy of a map, see (7), we see that if the refinement radius $r(N)$ is positive, we may control the geometric homotopy class by assuming an energy bound and sufficiently refining the graph Γ .

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